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Solitary waves in the Madelung's fluid: Connection between the nonlinear Schrödinger equation and the Korteweg-de Vries equation

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Abstract. An investigation to deepen the connection between the family of nonlinear Schrödinger equations and the one of Korteweg-de Vries equations is carried out within the context of the Madelung's fluid picture. In particular, under suitable hypothesis for the current velocity, it is proven that the cubic nonlinear Schrödinger equation, whose solution is a complex wave function, can be put in correspondence with the standard Korteweg-de Vries equation, is such a way that the soliton solutions of the latter are the squared modulus of the envelope soliton solution of the former. Under suitable physical hypothesis for the current velocity, this correspondence allows us to find envelope soliton solutions of the cubic nonlinear Schrödinger equation, starting from the soliton solutions of the associated Korteweg-de Vries equation. In particular, in the case of constant current velocities, the solitary waves have the amplitude independent of the envelope velocity (which coincides with the constant current velocity). They are *bright* or *dark* envelope solitons and have a phase linearly depending both on space and on time coordinates. In the case of an arbitrarily large stationary-profile perturbation of the current velocity, envelope solitons are *grey* or *dark* and they relate the velocity u_0 with the amplitude; in fact, they exist for a limited range of velocities and have a phase nonlinearly depending on the combined variable $x - u_0 s$ (s being a time-like variable). This novel method in solving the nonlinear Schrödinger equation starting from the Korteweg-de Vries equation give new insights and represents an alternative *key of reading* of the dark/grey envelope solitons based on the fluid language. Moreover, a comparison between the solutions found in the present paper and the ones already known in literature is also presented.

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1 Introduction

A number of problems in plasma physics are described in terms of suitable nonlinear Schrödinger equations (NLSE) [1–4]. Under suitable hypothesis, the well known Zakharov system of equations [5,6] gives a large variety of NLSEs describing the nonlinear wave propagation of e.m. structures [1], such as e.m. wavepackets and e.m. beams, as well as electrostatic structures, such as plasma wave envelopes. The large amplitude electromagnetic (e.m.) wave propagation in optical fibers and in transmission lines is governed by several type of NLSEs [7], as well.

A very important role is played by the NLSE also in mesoscopic physics, where it takes the names of Ginzburg–Landau [8,9] and Gross–Pitaevskij equations [10] which recently have been recognized to be very important in describing the Bose-Einstein condensation [10].

Additionally, within the framework of the quantum-like description provided by the Thermal Wave Model (TWM) [11] some kind of NLSE seem to be suitable for describing a number of problems in optics and in dynamics of charged-particle beams [12–14].

In all the above branches of researches, special attention has been devoted in literature for the envelope solitons of the several NLSEs, due to their peculiar characteristic of stability.

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On the other hand, a wide family of Kortweg-de Vries equations (KdVE) provides for the description of the nonlinear wave propagation in hydrodynamics (f.i., the shallow water waves) [15,6], gas-dynamics [15], plasma (f.i., the ion-acoustic wave propagation) [15], transmission line theory (f.i., lines with effect of the thermal noise) [16], and in charged-particle beam dynamics as well [17].

In particular, the investigations on the existence of solitons started more than one century ago just with the study of nonlinear wave propagation modelled with the classical KdVE [18]. Later on, it has been rapidly extended to the most of the above subjects, and at the present time the corresponding theories are supported by a large body of experimental evidences.

It is well known that the soliton satisfying the classical KdVE can be found with a very powerful method called *inverse scattering method* [19]. This method puts the KdVE in correspondence with a (linear) Schrödinger equation (LSE) in such a way that the soliton of the former plays the role of the (linear) potential of the latter. In this connection between KdVE and LSE, very important theorems have been found [20] and the inverse scattering has been soon applied to NLSE, as well [21]. Remarkably, the capability and the richness of similar methods currently applied to nonlinear partial differential equations for solving a number of physical problems have produced an autonomous research activity in mathematical physics usually called *inverse problems.*

In this paper, we want to extend the above connection between KdVE and LSE to the family of NLSE. Starting from the NLSEs, our analysis is carried out in the Madelug's fluid representation of an arbitrary NLSE. Consequently, in this framework, we will not solve an inverse problem. Under the hypothesis of stationary fluid, the main goals of our analysis are to show: (i) that we can transform one equation for the complex wave function into the other for the squared modulus of this wave function; (ii) the existence of envelope solitons of the cubic NLSE whose feature are the same of the classical KdVE solitons, which, in particular, relates the soliton velocity with its amplitude. This last property is not exhibited by the standard envelope solitons of the cubic NLSE for which the soliton amplitude is independent of the soliton velocity.

In the next section we formulate our problem and introduce the basic equation of the Madelung's fluid picture [22]. In Section 3, under the hypothesis that the current velocity is a given function, we find an equation for the density of the Madelung's fluid (*i.e.*, the squared modulus of the wave function associated with the NLSE) in the case of an arbitrary nonlinear potential. In Section 4, the current velocity is assumed constant; consequently, the above equation for the density describes nonlinear stationary states. Once the cubic nonlinearity for NLSE is assumed, it reduces to the classical KdV-type equation written for stationary-profile waves in terms of the combined independent variable (self-similar variable). Under this hypothesis, we find solitary solutions which exhibit the property, typical of solitary waves of NLSE, for which their velocity is independent of their amplitude. In Section 5, an analysis is carried out for the case in which all the variables of the Madelung's fluid, including the current velocity, are functions of the combined variable. In correspondence of a cubic nonlinearity of NLSE, the density equation reduces again to the classical form of KdVE and the system admits solitary waves fully similar to the standard Scott-Russel soliton of the shallow water. In particular, these solitons have the usual property of relating the velocity with the amplitude. Finally, conclusions and remarks are given in Section 6.

2 Formulation of the problem

Let us consider the following nonlinear Schrödinger-like equation (NLSE):

$$\mathrm{i}\alpha \frac{\partial \Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} + U\Psi, \tag{1}$$

where U is, in general, a functional of $|\Psi|^2$, *i.e.* $U = U\left[|\Psi|^2\right]$, the constant α accounts for the dispersive effects, and s and x are the time-like and the configurational coordinates, respectively. By representing Ψ as:

$$\Psi = \sqrt{\rho(x,s)}\exp\left[\frac{\mathrm{i}}{\alpha}\Theta(x,s)\right], \tag{2}$$

(note that $|\Psi|^2 = \rho$) it is well known that (1) is equivalent to the following pair of coupled fluid equations (Madelung's fluid [22]):

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0\,, \tag{3}$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right], \tag{4}$$

where the current velocity V is given by

$$V(x,s) = \frac{\partial \Theta(x,s)}{\partial x}\,\cdot \tag{5}$$

Note that, according to the above assumptions, $U = U[\rho]$.

Under suitable assumptions for the current velocity, we want to show that the set of equations (3, 4) (which is fully equivalent to Eq. (1)) can be cast in a nonlinear evolution equation for the density which has the form of a sort of generalized KdVE. The special hypothesis done for V will characterize the particular physical case considered. We will consider the following two different cases, only: (i) $V = V_0 =$ arbitrary constant; (ii) $V = V(\xi)$, where $\xi \equiv x - u_0 s$ is the combined variable (u_0 being in principle an arbitrary real constant). Provided that our attention is confined to consider the above two assumptions, the main goal of our study is to investigate on the existence of solitary solutions of the cubic NLSE and put them in correspondence with the ones of the standard KdVE.

3 Basic equations

In this section, starting from the system of equations (3, 4), we find an evolution equation for the density ρ, assuming that the current velocity $V(x,s)$ depends on x and s in a given way.

By multiplying equation (3) by V, the following equation can be obtained:

$$\rho\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V=-V\frac{\partial\rho}{\partial s}-V^2\frac{\partial\rho}{\partial x}+\rho\frac{\partial V}{\partial s}\cdot \tag{6}$$

Note that:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right)=\frac{1}{\rho}\left(\frac{1}{2}\frac{\partial^3\rho}{\partial x^3}-4\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}\right)\cdot \tag{7}$$

Furthermore, multiplying equation (6) by ρ and combining the result with (4) and (7) one obtains:

$$\rho\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V=-\frac{\partial U}{\partial x}\rho+\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}-2\alpha^2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}, \tag{8}$$

which combined again with equation (6) gives:

$$-V\frac{\partial\rho}{\partial s}-V^2\frac{\partial\rho}{\partial x}+\rho\frac{\partial V}{\partial s}= -\frac{\partial U}{\partial x}\rho+\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}-2\alpha^2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}\cdot \tag{9}$$

On the other hand, by integrating equation (4) with respect to x and multiplying the resulting equation by $\rho^{1/2}\left(\partial\rho^{1/2}/\partial x\right)$, we have:

$$-2\alpha^2\frac{\partial\rho^{1/2}}{\partial x}\frac{\partial^2\rho^{1/2}}{\partial x^2}=-2\frac{\partial\rho}{\partial x}\int\left(\frac{\partial V}{\partial s}\right)\mathrm{d}x -V^2\frac{\partial\rho}{\partial x}-2U\frac{\partial\rho}{\partial x}+2c_0(s)\frac{\partial\rho}{\partial x}, \tag{10}$$

where $c_0(s)$ is an arbitrary function of s. By combining (9) and (10) the following equation is finally obtained:

$$-\left(\frac{\partial V}{\partial s}\right)\rho+V\frac{\partial\rho}{\partial s}+2\left[c_0(s)-\int\left(\frac{\partial V}{\partial s}\right)\mathrm{d}x\right]\frac{\partial\rho}{\partial x} -\left(\frac{\partial U}{\partial x}\rho+2U\frac{\partial\rho}{\partial x}\right)+\frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3}=0. \tag{11}$$

In the next section, for the special cases mentioned in Section 2, equation (11) will be used to obtain the connection between NLSE (1) and a wide class of Korteweg-de Vries equation (KdVE) for the density ρ.

4 Motion with constant current velocity: Nonlinear stationary states

In this section, under the hypothesis of stationary Madelung's fluid, we show that the well known nonlinear envelope soliton solutions of the cubic NLSE can be easily recovered in the present formalism.

4.1 Equations for the stationary Madelung's fluid

Let us assume that the current velocity is an arbitrary constant:

$$V\equiv V_0. \tag{12}$$

Consequently, continuity equation (3) can be written as

$$\frac{\partial\rho}{\partial s}+V_0\frac{\partial\rho}{\partial x}=0, \tag{13}$$

which implies that ρ is a function of the combined variable $\xi\equiv x-V_0 s$, *i.e.*

$$\rho=\rho(\xi)=\rho(x-V_0 s). \tag{14}$$

It is easy to see directly from equation (4) that the physical assumption (12) implies $c_0(s)\equiv c_0=$ const., and in particular we have:

$$-\frac{\alpha^2}{2}\frac{\mathrm{d}^2\rho^{1/2}}{\mathrm{d}\xi^2}+U\rho^{1/2}=E\rho^{1/2}, \tag{15}$$

where

$$E=c_0-\frac{V_0^2}{2}=\text{const}. \tag{16}$$

The form of the equation (15) and the representation (2) imply that the corresponding eikonal is

$$\Theta(x,s)=V_0x-c_0s, \tag{17}$$

where we have used equation (5) together with equation (12). Consequently:

$$\Psi(x,s)=\rho^{1/2}(x-V_0s)\exp\left[\mathrm{i}kx-\mathrm{i}\omega s\right], \tag{18}$$

where $k\equiv V_0/\alpha$ and $\omega\equiv c_0/\alpha$. It is worth to emphasize that equation (15) is not the usual *linear* stationary Schrödinger equation. In fact, since U is an arbitrary functional of ρ, $U=U[\rho]$, equation (15) is in general a nonlinear equation, but nevertheless it describes a sort of *stationary states* in the configurational ξ-space (nonlinear eigenvalue problem).

Substituting (12, 14), and (16) in (11) we have:

$$2E\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\left(2U\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\mathrm{d}U}{\mathrm{d}\xi}\rho\right)+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0. \tag{19}$$

If one assumes that the functional U has the form:

$$U=q_0\rho^\beta, \tag{20}$$

with q_0 and β real constants, (19) becomes (stationary modified KdVE):

$$2E\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-(\beta+2)\,q_0\rho^\beta\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0. \tag{21}$$

4.2 Solitary waves of the cubic NLSE and their connection with the classical KdVE

For $\beta = 1$, equation (21) reduces to:

$$2E\frac{d\rho}{d\xi} - 3q_0\rho\frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} = 0, \tag{22}$$

which has the same form of the KdVE for wave solution with stationary profile. Then, equation (21) is the "natural" extension of this KdVE to a potential of form given by (20).

It is well known that, under suitable conditions, equation (22) admits both periodic (cnoidal waves) and localized solutions [20,15]. Provided that ρ is non-negative function, it is very important to observe that if ρ is a localized solution of (22), thus $\rho^{1/2}$ is a localized solution of (15).

Consequently, we can conclude that, for the physical case under discussion (*i.e.*, $V = V_0 = \text{const.}$), if Ψ is a solitary solution of the following cubic NLSE

$$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} + q_0|\Psi|^2\Psi, \tag{23}$$

thus $|\Psi|^2$ is a soliton solution of the following KdVE:

$$-\frac{2|E|}{|q_0|V_0}\frac{\partial\rho}{\partial s} - 3\rho\frac{\partial\rho}{\partial x} + \frac{\alpha^2}{4|q_0|}\frac{\partial^3\rho}{\partial x^3} = 0, \tag{24}$$

with $V_0 \neq 0$.

We can find the localized solutions of equation (22) according to the well known method presented in reference [15]. To this end we can consider two different boundary conditions.

4.3 Classical soliton with standard boundaries (bright soliton)

In case ρ satisfies the following boundary conditions in the ξ-space:

$$\lim_{\xi\to\pm\infty}\rho(\xi) = 0, \tag{25}$$

and provided that $q_0 < 0$, and $E < 0$, equation (22) has the following classical soliton solution:

$$\rho(x - V_0 s) = \rho_m \operatorname{sech}^2\left(\frac{x - V_0 s}{\Delta}\right), \tag{26}$$

where $\rho_m = 2|E|/|q_0|$, and $\Delta = |\alpha|/\sqrt{2|E|}$.

Correspondingly, by virtue of (18), the envelope soliton of the cubic NLSE (23) is given by:

$$\Psi(x,s) = \left(\frac{2|E|}{|q_0|}\right)^{1/2} \operatorname{sech}\left[\frac{\sqrt{2|E|}}{|\alpha|}(x - V_0 s)\right] \times \exp\left[\frac{i}{\alpha}\left[V_0 x - \left(E + V_0^2/2\right)s\right]\right], \tag{27}$$

where we have used equation (16). Solutions (26) and (27) describe standard KdVE soliton and NLSE envelope soliton, respectively. Note that

$$\rho^{1/2}(x - V_0 s) = \left(\frac{2|E|}{|q_0|}\right)^{1/2} \operatorname{sech}\left[\frac{\sqrt{2|E|}}{|\alpha|}(x - V_0 s)\right] \tag{28}$$

is a solution of the nonlinear-stationary-state equation (15), when $U[\rho]) = q_0\rho$. Note also that Δ and ρ_m satisfy the following property:

$$\Delta^2\rho_m = \frac{\alpha^2}{|q_0|} = \text{const.}, \tag{29}$$

and ρ_m is independent of the soliton velocity V_0.

For $V_0 = 0$, (27) becomes a nonlinear localized stationary state of the cubic NLSE (23) with $q_0 < 0$ and $E < 0$.

4.4 Dark solitons

We now consider the cases for which $\lim_{\xi\to\pm\infty}\rho(\xi) \equiv \rho_0 > 0$. Thus, we can cast ρ as:

$$\rho(\xi) \equiv \rho_0 + \rho_1(\xi). \tag{30}$$

Consequently, the following equation for ρ_1 can be written:

$$2E'\frac{d\rho_1}{d\xi} - 3q_0\rho_1\frac{d\rho_1}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho_1}{d\xi^3} = 0, \tag{31}$$

where $E' \equiv E - 3q_0\rho_0/2$. Then, we assume the following boundary condition for ρ_1:

$$\lim_{\xi\to\pm\infty}\rho_1(\xi) = 0,$$

which implies, by virtue of (15), that

$$E = q_0\rho_0 \tag{32}$$

and, consequently

$$E' = -\frac{1}{2}q_0\rho_0. \tag{33}$$

If we look for solutions corresponding to $\rho_1 > 0$ (bright solitons), it is easily seen that no soliton solutions can be found. However, soliton solutions exist when

$$q_0 > 0 \quad \text{and} \quad E' < 0, \tag{34}$$

which corresponds to $\rho_1 < 0$ (*dark* soliton). In this case, in fact, equations (32, 33) and (34) are fully consistent. Consequently, provided that $|\rho_1| \leq \rho_0$ to keep ρ non-negative, we can conclude that *dark solitons are possible.* Thus, solving equation (31) for ρ_1 and using equation (30) we easily find the following soliton solution for ρ:

$$\rho(x - V_0 s) = \rho_0 \tanh^2\left[\frac{\sqrt{q_0\rho_0}}{|\alpha|}(x - V_0 s)\right]\cdot \tag{35}$$

Correspondingly, according to equation (18), we obtain the following envelope soliton of the cubic NLSE:

$$\Psi(x,s)=\sqrt{\rho_0}\left|\tanh\left[\frac{\sqrt{q_0\rho_0}}{|\alpha|}(x-V_0 s)\right]\right| \times \exp\left\{\frac{\mathrm{i}}{\alpha}\left(V_0 x-\left(q_0\rho_0+\frac{V_0^2}{2}\right)s\right)\right\}, \quad (36)$$

where now $c_0 = q_0\rho_0 + V_0^2/2$. Note that here $\Delta' \equiv |\alpha|/\sqrt{q_0\rho_0}$ and $\rho_m' \equiv -\rho_0$ are soliton's width and soliton's minimum amplitude, respectively; they satisfy the following property, fully similar to (29):

$$\left(\Delta'\right)^2\left|\rho_m'\right|=\frac{\alpha^2}{|q_0|}=\text{const.} \quad (37)$$

Note that, for $V_0 = 0$, (36) reduces to a nonlinear localized stationary state of the cubic NLSE (23) with $q_0 > 0$ and $E = q_0\rho_0 > 0$.

Notice also that the modulus in (36), which was missing in equation (1) of the second paper of reference [23], is essential to provide a soliton solution of the NLSE.

We would like to stress that all the solitary waves found, in this section, for the cubic NLSE have an amplitude-independent velocity.

5 Motion with stationary-profile current velocity

In this section we assume that both the quantity ρ and V involved in the Madelung's fluid equations (3, 4) are function of the combined variable $\xi \equiv x - u_0 s$, u_0 being a real constant, *i.e.*

$$\rho(x,s)=\rho(\xi), \quad \text{and} \quad V(x,s)=V_0+V_1(\xi), \quad (38)$$

where V_0, as in the previous section, is an arbitrary constant current velocity associated with the Madelung's fluid background motion and $V_1(\xi)$ is an arbitrarily large current velocity perturbation.

5.1 Basic equations

According to the hypothesis (38), equation (3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\rho V)=u_0\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\cdot \quad (39)$$

Equation (39) can be easily integrated, giving:

$$V_0+V_1=u_0+\frac{A_0}{\rho}, \quad (40)$$

where A_0 is an arbitrary constant. Additionally, it is also easy to see that the arbitrary function of s $c_0(s)$ appearing in equation (10) (it comes from the integration equation (4) with respect to x) becomes constant in the present case. This constant accounts for the energy conservation.

Consequently, equation (11) becomes now:

$$\left(2c_0+u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-\left(2U\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\mathrm{d}U}{\mathrm{d}\xi}\rho\right)+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0, \quad (41)$$

which is very similar to equation (19). Analogously, assuming that the functional U has the form (20), (41) becomes

$$\left(2c_0+u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-(\beta+2)\,q_0\rho^\beta\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0, \quad (42)$$

which is very similar to the generalized KdVE (21).

5.2 Grey solitary solutions

We observe that, for the physical case under discussion, the boundary conditions (25) cannot be applied due to equation (40). In fact, as $\xi \to \pm\infty$, $V(\xi)$ would diverge. Consequently, we have to impose the following boundary conditions for ρ:

$$\lim_{\xi\to\pm\infty}\rho(\xi)=\rho_0, \quad (43)$$

where ρ_0 is a positive constant. It follows that V satisfies the following boundary conditions:

$$\lim_{\xi\to\pm\infty}V_1(\xi)=0. \quad (44)$$

Consequently, the continuity equation (40) gives us:

$$A_0=-\rho_0\left(u_0-V_0\right). \quad (45)$$

The case of standard KdVE is obtained from equation (42) for $\beta = 1$, namely:

$$\left(2c_0+u_0^2\right)\frac{\mathrm{d}\rho}{\mathrm{d}\xi}-3q_0\rho\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0. \quad (46)$$

By putting:

$$\rho(\xi)=\rho_0+\rho_1(\xi), \quad (47)$$

from (46) we obtain:

$$2E''\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}-3q_0\rho_1\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}+\frac{\alpha^2}{4}\frac{\mathrm{d}^3\rho_1}{\mathrm{d}\xi^3}=0, \quad (48)$$

where $2E'' = \left(2c_0 + u_0^2 - 3q_0\rho_0\right)$. Note that, boundary conditions (43) and (44) implies that $c_0 = q_0\rho_0$ and, consequently:

$$2E''=\left(u_0-V_0\right)^2-q_0\rho_0.$$

Of course, ρ_1 satisfies boundary conditions:

$$\lim_{\xi\to\pm\infty}\rho_1(\xi)=0. \quad (49)$$

As in the previous section, we can consider the two different cases of $\rho_1 > 0$ and $\rho_1 < 0$, respectively.

If $\rho_1 > 0$ (bright solitons, equation (48) would have soliton solution for

$$q_0 < 0 \quad \text{and} \quad E'' < 0. \tag{50}$$

However, according to E'' definition, we have:

$$2E'' = (u_0 - V_0)^2 + |q_0|\rho_0 > 0, \tag{51}$$

which contradicts the second of (50). Consequently, *no bright solitons are possible*.

If, however, $\rho_1 < 0$, corresponding to *grey* solitons, equation (48) admits solutions for

$$q_0 > 0 \quad \text{and} \quad E'' < 0, \tag{52}$$

provided that the condition $|\rho_1| \leq \rho_0$ is satisfied. In particular, the second of (52) implies that:

$$(u_0 - V_0)^2 \leq q_0\rho_0, \tag{53}$$

namely

$$V_0 - \sqrt{q_0\rho_0} \leq u_0 \leq V_0 + \sqrt{q_0\rho_0}. \tag{54}$$

In analogy with Section 5.2, equation (48) would have the following solution:

$$\rho_1(\xi) = -\frac{q_0\rho_0 - (u_0 - V_0)^2}{q_0}\,\text{sech}^2\left[\frac{\sqrt{q_0\rho_0 - (u_0 - V_0)^2}}{|\alpha|}\xi\right] \tag{55}$$

and the solution of (15), when $U[\rho] = q_0\rho$, is:

$$\rho(\xi) = \rho_0\left[1 - A^2\,\text{sech}^2\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)\right], \tag{56}$$

where

$$A^2 = \frac{q_0\rho_0 - (u_0 - V_0)^2}{q_0\rho_0} \geq 0. \tag{57}$$

Note that: $A^2 \leq 1$, which implies $-1 \leq A \leq 1$. Let us observe that, from (55), condition $|\rho_1| \leq \rho_0$ together with (53) imply that $(u_0 - V_0)^2 \geq 0$, which proves the consistency of the above conditions.

By combining equations (40, 56) the solution for $V_1(\xi)$ is obtained as:

$$V_1(\xi) = -\frac{(u_0 - V_0)\,A^2\,\text{sech}^2\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)}{1 - A^2\,\text{sech}^2\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)}\cdot \tag{58}$$

However, in order to construct the solution for Ψ by using equation (2), we now use equation (5), *i.e.*

$$V_0 + V_1(\xi) = \frac{\partial\Theta_0(x,s)}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}\xi}\Theta_1, \tag{59}$$

where we have splitted $\Theta(x,s)$ in two parts, *i.e.*

$$\Theta(x,s) = \Theta_0(x,s) + \Theta_1(\xi). \tag{60}$$

Then, by assuming $V_0 = \partial\Theta_0/\partial x$ and $V_1 = \partial\Theta_1/\partial\xi$, we can easily integrate for Θ_0 and Θ_1, respectively, obtaining

$$\Theta_0(x,s) = V_0 x - \left(q_0\rho_0 + \frac{V_0^2}{2}\right)s \tag{61}$$

and:

$$\Theta_1(\xi) = \Theta_{10} - \frac{(u_0 - V_0)\,|\alpha| A}{\sqrt{q_0\rho_0 A^2\,(1 - A^2)}} \times \tan^{-1}\left[\frac{A}{\sqrt{1 - A^2}}\tanh\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)\right], \tag{62}$$

where Θ_{10} is an arbitrary constant.

Consequently, the solution of the cubic NLSE, *i.e.* Ψ, is:

$$\Psi(x,s) = \sqrt{\rho(\xi)}\exp\left[\frac{\mathrm{i}}{\alpha}\Theta(\xi)\right], \tag{63}$$

where $\rho(\xi)$ and $\Theta(\xi)$ are given by (56, 61) and (62), respectively. Note that, for $u_0 - V_0 \neq 0$, this solution represents a *grey* envelope soliton (*i.e.* the minimum amplitude does not reach zero), whilst, for $u_0 - V_0 = 0$, it represents a dark soliton (the minimum amplitude is zero). In fact, as it can be seen by (57), $u_0 - V_0 = 0$ corresponds to $A^2 = 1$ (and *vice versa*) and, from (45), that $A_0 = 0$. Consequently, as it can be also seen from (40), V is identical to V_0 and V_1 vanishes. Hence, this dark soliton solution is nothing else but the dark soliton found in Section 4. Thus, the present section has extended the dark soliton concept to the grey solitons by means of the inhomogeneity of the current velocity $V(\xi)$.

We can conclude that in the case of stationary-profile solutions, both grey and dark solitons exist with non-arbitrary velocity. In fact, u_0 must satisfy the condition (54). Figures 1, 2 and 3 show the grey envelope soliton in terms of $|\Psi| = \sqrt{\rho(\xi)}$, $V_1(\xi)$ and $\Theta_1(\xi)$, respectively.

Note that, according to the above results, our grey envelope soliton is given for arbitrary values of the parameter α. Furthermore, by using (57) and (62), we have

$$\frac{\Theta_1(\xi)}{\alpha} = \frac{\Theta_{10}}{\alpha} - \text{sign}\,(u_0 - V_0)\,\text{sign}\,(\alpha) \times\tan^{-1}\left[\frac{\sqrt{q_0\rho_0 - (u_0 - V_0)^2}}{|u_0 - V_0|} \times\tanh\left(\frac{\xi\sqrt{q_0\rho_0 - (u_0 - V_0)^2}}{|\alpha|}\right)\right]\cdot \tag{64}$$

Consequently, from (56, 61) and (64) one can easily see how the envelope solution (63) depends on a sign change

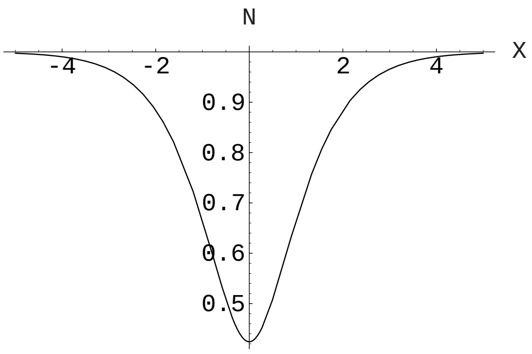

Fig. 1. $N \equiv |\Psi|/\sqrt{\rho_0} = \sqrt{\rho/\rho_0}$ *versus* $X \equiv \xi/|\alpha|$ where $\xi = x - u_0 s$, for $\rho_0 q_0 = 0.5$, $u_0 = 1$, and $V_0 = 0.7$.

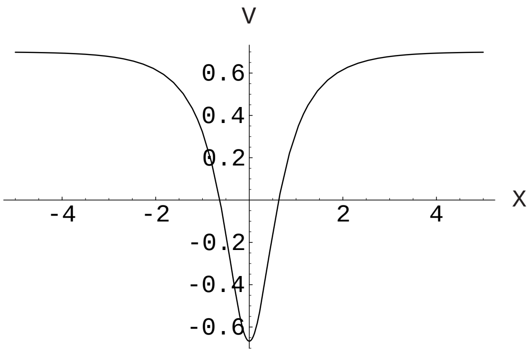

Fig. 2. V *versus* $X \equiv \xi/|\alpha|$, where $\xi = x - u_0 s$, for $\rho_0 q_0 = 0.5$, $u_0 = 1$, and $V_0 = 0.7$.

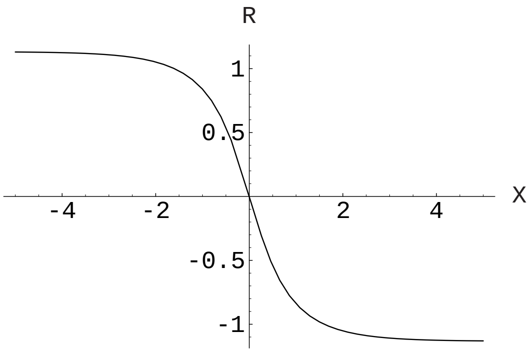

Fig. 3. $R \equiv \Theta_1/\alpha$ *versus* $X \equiv \xi/|\alpha|$, where $\xi = x - u_0 s$, for positive α, $\rho_0 q_0 = 0.5$, $u_0 = 1$, $V_0 = 0.7$, and $\Theta_{10} = 0$.

of α and $u_0 - V_0$, respectively. In particular, note that a sign change of α does not change $|\Psi|$. Finally, note that, since the following identity

$$\tan^{-1}\left[\frac{A}{\sqrt{1-A^2}}\tanh\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)\right] = \sin^{-1}\left[\frac{A\tanh\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)}{\sqrt{1-A^2\,\mathrm{sech}^2\left(\frac{\sqrt{q_0\rho_0 A^2}}{|\alpha|}\xi\right)}}\right] \quad (65)$$

holds, the envelope soliton found above recovers, for nonzero u_0 and V_0 and such that $u_0 - V_0 \neq 0$, the one found in reference [23], in which only the special case of $\alpha < 0$ (namely, $\omega_0^{"} < 0$ in Ref. [23]) has been considered. In particular, our solutions, as results of the above new method in solving NLSE starting from the associated KdVE, give new insights and an alternative *key of reading* of the envelope dark/grey solitons of the cubic NLSE in terms of a fluid language (see, f.i., the soliton solution for the current velocity), which is usual for soliton solutions of the KdVE.

Finally, we notice that the present soliton solution within the framework of the Madelung fluid, which in the unperturbed state moves with the constant velocity V_0, can be equally well obtained by a Galilean transformation from the corresponding soliton solution based on a Madelung fluid that in the unperturbed state has no background flow.

6 Conclusions and remarks

In this paper, we have presented an investigation for the existence of solitary waves of NLSE that has been carried out within the framework of the Madelung's fluid picture. We have considered the case of constant current velocity ($V = V_0$) as well as the one of arbitrarily large amplitude perturbation of the current velocity with stationary-profile ($V = V_0 + V_1(\xi)$). In both cases we have shown that the pair of motion and continuity equations (which are fully equivalent to NLSE) can be transformed into a suitable KdVE for waves with stationary profiles. If localized solutions are requested, the first case ($V = V_0$) recovers the well known envelope solitons of the NLSE whose main feature is that the amplitude is independent of the velocity V_0. This family contains standard NLSE envelope solitons which are *bright* or *dark* whose phase is linear in x and s. The squared modulus of these envelope solitons is a solution of a suitable KdVE. The second case ($V = V_0 + V_1(\xi)$) recovers localized waves that relate the amplitude with the velocity u_0 and have a nonlinear phase. Also in this case, we have shown that the squared modulus of these envelope solitons are solutions of a suitable KdVE. However, we have found that, in this family of solutions, the envelope solitons are *grey* and they exist for a limited range of u_0 (see Eq. (54)). Moreover, we have shown that the existence of this grey solitons family is strictly connected with the soliton-like inhomogeneity of the current velocity ($V(\xi)$). This result is a peculiarity of the Madelung's fluid description and, therefore, is not fully evident in the envelope soliton solution of the cubic NLSE as given in reference [23]. We have also shown that the limiting case of *dark* solitons ($A^2 = 1$) recovers the corresponding solution obtained in Section 4 ($V = V_0$). Thus we have proven that a correspondence between the cubic NLSE and the usual KdVE exists, within which a soliton solution of the latter is the squared modulus of the envelope soliton of the former.

We would like to point out that the method used in this paper to provide localized solutions for the NLSE within the context of the Madelung's fluid picture and to get a

KdVE equation seems to be powerful and promising. Furthermore, it provides for an alternative physical interpretation, based on a fluid picture, of the nonlinear problems related to NLSE in a number of different physical situations, such as nonlinear propagation of e.m. wavepackets in optical fibers [7] and plasmas (f.i., Langmuir solitons) [15,6]. In the longitudinal charged-particle beam dynamics of the circular accelerating machines it may be important to investigate on possible connections between standard solitons, predicted by the standard fluid and kinetic theories [17,24], and envelope solitons, predicted by the Thermal Wave Model [12,13]. In mesoscopic physics it may give new insights in the formation mechanisms of nonlinear localized structures predicted by the theory of Ginzburg and Landau [8,9] and by the Gross-Pitaevskij equation [10].

A natural extension of the present work would be the study of the case corresponding to the current velocity as an arbitrary function of the time-like variable, s. This physical assumption would allow us to describe in the Madelung's fluid picture the coherent motion of nonlinear structures. It is clear from equation (11) that a sort of generalized KdVE can be written for the squared modulus of the NLSE wave function. In perspectives, an investigation on the existence of solitary waves for such a kind of equation will be carried out in a future work.

One also can think to get multi-soliton solutions in the standard theory of KdVE (*e.g.* by the inverse scattering method or by the Hirota's method [25]). Thus, nonstationary-profile solutions must be looked for. However, in our method we are dealing with stationary-profile solutions only. Multi-soliton solutions are, hence, out of the scope of the present paper. Nevertheless, by allowance of a more general dependence between ρ and V (which have to satisfy the system of coupled Eqs. (3) and (11)) and by specifying the explicit form of the functional $U[\rho]$, one could imagine to find multi-solitons also for the Madelung's fluid.

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